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Introduction.* Following Curry we distinguish two kinds of systems, assertional and relational. The assertional systems contain either a unary predicate for which we use the assertion sign '⊢', or expressions such as '______ is provable', or '______ is in T'. Relational systems contain a binary relational predicate. Both systems are equivalent in the sense that statements of one system can be translated into statements of the other preserving truth.

The system we are going to develop will be relational, and the basic relation will be an equality, as in the systems of ordinary mathematics. The basic axioms will be common to our system and to ordinary arithmetic. The addition of two more axioms will give our system its specific features which give it logical interest, and allow for a method of rendering complicated formulae of propositional calculus in a simpler way, in line with our algebraic intuitions. On the other hand, the resulting set of axioms is of interest, for its divergence from traditional ones.

We will develop four systems **S**, **H**, **I**, and **L** with three basic operations. The system **S**, with axioms, will be an arithmetical system twelve which is true for positive numbers for instance, when the three basic operations are interpreted as the arithmetical ones. The addition of one axiom will lead us to the second system **H**, a distributive lattice. The third system **I**, through the addition of another axiom, can be interpreted as the intuitionist propositional logic, and a last condition will give us a system interpretable as classic bivalent logic, or as a calculus of classes.

1. *System* **S**. We postulate the existence of a set S of elements $a, b, c \ldots$ with two distinguished elements 1 and 0 closed under three binary operations $a + b$, $a . b$, b^a, satisfying the following conditions[1] for every $a, b, c \in S$.

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A1 $(a + b) + c = a + (b + c)$
A2 $(a . b) . c = a . (b . c)$
A3 $a + b = b + a$
A4 $a . b = b . a$
A5 $a(b + c) = ab + ac$
A6 $(ab)^c = a^c . b^c$
A7 $a^{(b+c)} = a^b . a^c$
A8 $(a^b)^c = a^{bc}$
A9 $a + 0 = a$
A10 $a . 1 = a$
A11 $a^1 = a$

We define now in the set S a binary relation $\leq$ and we add an antisymmetry axiom.

D1 $a \leq b$ if and only if there exists an $x \epsilon S$, such that $b = a + x$
A12 $a \leq b$ *and* $b \leq a$ *imply* $a = b$

We can easily prove that the relation $\leq$ has the properties of reflexivity and transitivity, and define a partial order in **S** with 0 as the unique minimal element. We shall derive some interesting consequences which will be useful also in further proofs.

T101	$a \leq a$	by *A9, D1*
T102	$a \leq b$ *and* $b \leq c$ *imply that* $a \leq c$	by *A1, D1*
T103	$0 \leq a$ *for every* $a \epsilon S$	by *A9, D1*
T104	$a \leq b$ *implies* (1) $a + c \leq b + c$ *and* (2) $a . c \leq b . c$	by *D1, A2, A3*
		by *D1, A5*
T105	$a \leq b$ *and* $c \leq d$ *imply* (1) $a + c \leq b + d$ (2) $ac \leq bd$	by *D1, A2, A4*
		by *D1, A5, A2*
T106	$(a^b)^c = a^{bc} = (a^c)^b$ (Importation-Export. laws)	by *A8, A4*

2. *The System* **H**. To the conditions that define **S**, we add the following axiom, for every $a \epsilon S$.

A13 $a \leq 1$

The addition of *A13* gives us immediate interesting consequences

T201	$a . 0 = 0$	
PR	$0 . a \leq 0 . 1 \leq 0$	*A3, T103*
	$a . 0 = 0 . a = 0$	*T105, A12*
T202	$1 + a = 1$	
PR	$1 \leq 1 + a \leq 1$	*D1, A13*
	$1 + a = 1$	*A12*
T203	$a + a = a$ (Idempotent or Tautology law)	
PR	$a + a = a(1 + 1) = a . 1 = a$	*A10, T202, A5*
T204	$a . a = a$	
PR	$a^1 = a^{1+1} = a . a = a$	*A11, T202, A7*
T205	$ab \leq a$ (A fortiori law)	*T101, A13, T104*

T203, T204, T205 and the monotony laws, *T104* and *T105*, imply that the System **H** is a lattice ordered by $\leq$. By *A5*, it is a distributive one.

T206 $a \leq b$ *implies* (1) $a + b = b$ *and* (2) $a . b = a$ — *D1, T203; A5, T205, T206*

In the next lines we shall bring some theorems on exponentiation which are true in **H**.

T207 $b \leq c$ *implies* (1) $a^c \leq a^b$; (2) $b^a \leq c^a$ — *D1, A7, T205; T206, A6, T205*
T208 $1^a = 1$ — *A13, A11, T207, A12*
T209 $a \leq a^b$ (Verum Sequitur ad quodlibet) — *A13, T207*
T210 $a^b \leq (a^c)^b$ (Interpolation law) — *T209, T207*
T211 $(a^b)^b = a^b$ — *T209, T210, A12*
T212 $b^a . d^c \leq bd^{ac}$ (Praeclarum Theorema) — *T209, T104, A8, A6*

3. *The System* **I**. We add to **H** the following condition, for every $a, b \in S$

A14 $a^b = 1$ if and only if $b \leq a$.

Then:

T301 $a^0 = 1$ — *T105, A14*
T302 (1) $b . a^b \leq a$ (Assertion law)

PR $(a^b)^{(a^b)} = 1$ — *A14, T101*
$a^{b \cdot a^b} = 1$ — *A8*
$b . a^b\ a$ — *A14*

and

(2) $ba^b = ab$ — *T209, T103, T205, T302, T104, A12*
T303[2] $ab \leq c$ *iff* $a \leq c^b$ — *A14, A8*
T304 $[(ab)^a]^b = 1$
T305 $c^b . b^a \leq c^a$ (Syllogistic law) — *T302, T303*
T306 $b^a \leq (c^a)^{(c^{ab})}$ — *T302, T207, A6, T303*
T307 $b^a \leq bc^{ac}$ — *T302, T103, T303*
T308 $b^a \leq (b + c)^{(a + c)}$

We will now define negation in the following way
D2 $\overline{a} =_{df} 0^a$
T309 $\overline{a} \leq b^a$ — *T207, D2*

T309 proves that the System **I** is the propositional intuitionistic system where $\overline{a}$ is to be interpreted as $\neg a$.[3]

T310 $a \leq b^{\overline{a}}$ (Ex Falso sequitur quodlibet) — *T309, T303*
T311 $b^a \leq \overline{a}^{\overline{b}}$

PR $0^b . b^a \leq 0^a$ — *T305*
$b^a \leq (0^a)^{(0^b)}$ — *T303*

T312[4] $\overline{b}^a \leq \overline{a}^b$ — *D2, T106*
T313 $\overline{0} = 1$ and $\overline{1} = 0$ — *D2, T301; D2, A11*
T314 $a . \overline{a} = 0$ (Principle of Non-Contradiction) — *D2, T303*
T315 $a \leq \overline{\overline{a}}$ — *T314, T303*

The converse is not provable in System **I**.

T316	$\overline{a} = \overline{\overline{\overline{a}}}$	*T315, T207, A12*
T317	$\overline{a} \leq \overline{ab}$	*T205, T207*
T318	(1) $\overline{a} + \overline{b} \leq \overline{ab}$ (De Morgan laws)	
	(2) $\overline{a} + b \leq \overline{a\overline{b}}$	
	(3) $a + \overline{b} \leq \overline{\overline{a}b}$	

The converses are not theorems of **I**.

(4) $\overline{a + b} \leq \overline{a} . \overline{b}$

(5) $ab \leq \overline{\overline{a} + \overline{b}}$

$\overline{a}b \leq \overline{a + \overline{b}}$

$a\overline{b} \leq \overline{\overline{a} + b}$

$\overline{a} . \overline{b} \leq \overline{a + b}$

T319	$\overline{\overline{a + \overline{a}}} = 1$	*D2, A7, T302, T313*
T320	$b^{a} . \overline{b}^{a} \leq \overline{a}$ (Reductio ad absurdum)	*T306, T303*

4. *System* **L**. To the axioms of system **I** we add an Involution Axiom for negation. Similar results could be obtained if we restrict ourselves to elements that can be written as negation of other elements, and use *T316*. System **L** can thus be interpreted as the classical logical system.

A15 $\overline{\overline{a}} = a$

Then:

T401	$\overline{ab} = \overline{a} + \overline{b}$	*A7, A15*
T402	$a^{b} = b . \overline{a}$	*A8, A15*
T403	$a^{b} = a + \overline{b}$	*T401, T402*
T404	$a + \overline{a} = 1$ (Tertium non datur)	*A13, T403*
T405	(1) $a^{\overline{b}} = b^{\overline{a}}$ and (2) $\overline{b}^{\overline{a}} = a^{b}$	*T403, A15; T403, A3*
T406	$a = a(b^{a})$ (Pierce's law)	*T403, T402, T205, T206, T209*
T407[5]	$a^{b} + b^{a} = 1$	*T404, T209, T309*

NOTES

1. For an easier understanding of the logical application of the systems, let us propose the following propositional and class interpretation, the first one in Peano-Russell's notation

$a + b$	$a \vee b$	$\alpha \cup \beta$
$a \cdot b$	$a \wedge b$	$\alpha \cap \beta$
b^{a}	$a \supset b$	$\hat{x}(x \in \alpha \supset x \in \beta)$
$a = b$	$\vdash a \equiv b$	$\alpha = \beta$
$a \leqslant b$	$\vdash a \supset b$	$\alpha \subset \beta$
$a = 1$	$\vdash a$	$\alpha = V$
$a = 0$	$\vdash \neg a$	$\alpha = \Lambda$

In the propositional calculus 1 and 0 can be diversely interpreted. For a full development of the different possibilities see Curry, [1], chap. 6.

For the nature of the basic relation =, see Curry, [1] chap. 3, especially p. 101-105.

2. For this theorem *see* Curry [1], p. 140. The assertion law *T302* may be seen as a simplification rule:

$$a \,.\, b^{a} \leqslant \not{a} \,.\, b^{\not{a}} \leqslant b$$

T303 may be seen as a procedure useful for conditional proof, to bring elements from one member to another in an inequation.

3. *See* Kleene [2], p. 82 and 101.

4. It is suggestive to point out that other contraposition laws are unprovable in the System **I**. For instance $b^{\bar{a}} \leqslant a^{\bar{b}}$ is not a theorem. The original form has 0^{a} in the exponent.

5. Negation can be seen in system **L** as a kind of logarithm to the base 0, where logarithmic rules are as usual. For instance, starting with $\vdash a \supset a$ we get:

$$a^{a} = 1 \qquad \lg.\,(a^{a}) = \lg.\,1 \qquad a\,.\,\lg.\,a = 0$$

Hence

$$a \,.\, \bar{a} = 0 \quad \text{and}$$
$$\bar{a} + a = 1$$

De Morgan laws are instances of this procedure.

These laws and the rules of note 2 give us a simple and mechanical procedure for deduction.

Let us see a simple exercise taken from P. Suppes' *Introduction to Logic*, p. 29.

1. $C \supset (D \supset B)$
2. $\sim G \vee C$
3. D
4. G

$\vdash G \supset B$

$(B^{D})^{C} \,.\, (\bar{G} + C) \,.\, D.G \leqslant B^{DC} \,.\, C^{\not{G}} \,.\, D.\not{G} \leqslant B^{\not{D}\not{C}} \,.\, \not{D}\not{C} \leqslant B \leqslant B^{G}$

Hence $G \supset B$.

It seems to me that the pedagogical implications of this notation and procedure are of special interest, rendering complicated formulae in a simpler way more in line with our algebraic intuitions.

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[2] Kleene, S. C., *Introduction to Metamathematics*, Amsterdam (1967).

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